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Degravitation Features in the Cascading Gravity Model

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Abstract

We obtain the effective gravitational equations on the codimension-2 and codimension-1 branes in the cascading gravity model. We then apply our formulation to the cosmological case and obtain the effective Friedmann equations on the codimension-2 brane, which are generically given in terms of integro-differential equations. Adopting an approximation for which the thickness of the codimension-2 brane is much smaller than the Hubble horizon, we study the Minkowski and de Sitter codimension-2 brane solutions. Studying the cosmological solutions shows that the cascading model exhibits the features necessary for degravitation of the cosmological constant.

1 Introduction

The cosmological constant problem is one of the most pressing conceptual problems in physics. This problem arises because the observed value of the vacuum energy is very small as compared to the values inferred from quantum field theory. Recently, a braneworld model which could provide a promising framework for addressing the cosmological constant problem has been developed. This model is the so-called cascading gravity model [1, 2, 3, 4], which could induce infrared modifications of gravity that can screen the effect of a cosmological constant. This idea, referred to as 'degravitation' [1], could possibly provide a dynamical solution to the cosmological constant problem, since any large cosmological constant initially present would degravitate away over time.

In the cascading gravity model which is a generalization of the Dvali-Gabadadze-Porrati (DGP) model [5] to higher dimensions, one constructs a sequence of branes with decreasing dimensions placed one on each other, where each brane action contains the induced gravity term. The gravitational force falls off faster in large distances in the cascading model than in the original five-dimensional (5D) DGP model [2]. In the simplest six-dimensional (6D) cascading model, our 4D Universe, codimension-2 brane, is placed on a codimension-1 brane which is embedded into a 6D bulk spacetime. The 6D model contains two cross-over scales, $r_3 := \frac{M_4^2}{M_5^3}$ and $r_4 := \frac{M_5^3}{M_6^4}$, where M_6, M_5 and M_4 are gravitational energy scales in the bulk and on the codimension-1 and codimension-2 branes, respectively. Assuming that $r_3 \ll r_4$, it is expected that the gravitational potential on the codimension-2 brane cascades from the 4D regime at short scales, to the 5D one at intermediate distances and finally to the 6D regime at large distances. This model addresses a problem in the 6D brane world models with the induced gravity. If there is no induced gravity term on the codimension-1 brane, the bulk graviton propagator diverges logarithmically near the position of the codimension-2 brane. Then the energy scale r_4^{-1} acts as an infrared cut-off for the propagator [4] so that it remains finite even at the position of the codimension-2 brane. The cascading model suffers a ghost instability if there is no tension on the codimension-2 brane.

However, very interestingly, it has been shown that there is a critical tension above which the model becomes ghost free [4, 6]. This model may also provide a mechanism of the degravitation [2, 3] which can support a very small expansion rate of our Universe even in the presence of a large cosmological constant, namely the codimension-2 brane tension. Hence, the degravitation could provide a way to resolve the cosmological constant problem.

A purpose of this paper is to see whether in reality there could be some features of degravitation in the cascading gravity model. In order to establish if this model exhibits degravitation it is necessary to understand its cosmological evolution and obtain the effective Friedman equations. The cosmological behavior can be very different from the ordinary 4D cosmology and the 5D DGP model, although they should be recovered in a certain limit. Cosmology in the cascading gravity model has been studied in the context of the 5D theory in [7], which is composed of gravity coupled to a scalar field [8], originated from the bending of the codimension-1 brane in the 6D bulk. This theory is a nonlinear extension of the weak gravity limit of the full 6D model, but it is not the unique extension. Although the 5D theory may possess several similarities to the full 6D theory, the final confirmation should be made in the context of the original 6D theory. A regular solution involving a flat codimension-2 brane with a tension has also been discussed in [9]. Some implications of the cascading gravity model to cosmological events have been discussed in, e.g., Ref. [10, 11].

In this paper, we present the covariant formulation of the nonlinear effective gravitational theory on the boundaries, codimension-1 and codimension-2 branes. We then apply them to find the cosmological solutions, in particular the de Sitter solutions which may possess features of degravitation. In the gravitational equations on the codimension-2 brane, the bulk contributions are given in terms of the integration over the sixth direction where we take a finite thickness of the codimension-2 brane into consideration. We study some cosmological solutions that show degravitation of the cosmological constant in the cascading model. In these solutions the effect of the effective vacuum energy into the Hubble expansion rate is suppressed by the ratio $\frac{r_3}{r_4}$ in comparison with the naive expectation from the ordinary 4D cosmology. Moreover, we will discuss some self-accelerating solutions by applying the small thickness approximations where the codimension-2 brane's thickness is assumed to be much smaller than the size of the Hubble horizon.

2 Gravitational equations in the cascading gravity model

In the simplest realization of the cascading gravity our codimension-2 brane world is placed on a codimension-1 brane embedded into a 6D bulk space-time. We start from the general ADM form of the 6D metric, with the bulk coordinate y playing the role of a time variable [1]

$$^{(6)}ds^2 = N^2 dy^2 + g_{ab}(dx^a + N^a dy)(dx^b + N^b dy), \tag{1}$$

[1] The 6D metric is

$$g_{AB} = \begin{pmatrix} N^2 + N^a N_a & N_a \\ N_a & g_{ab} \end{pmatrix}.$$

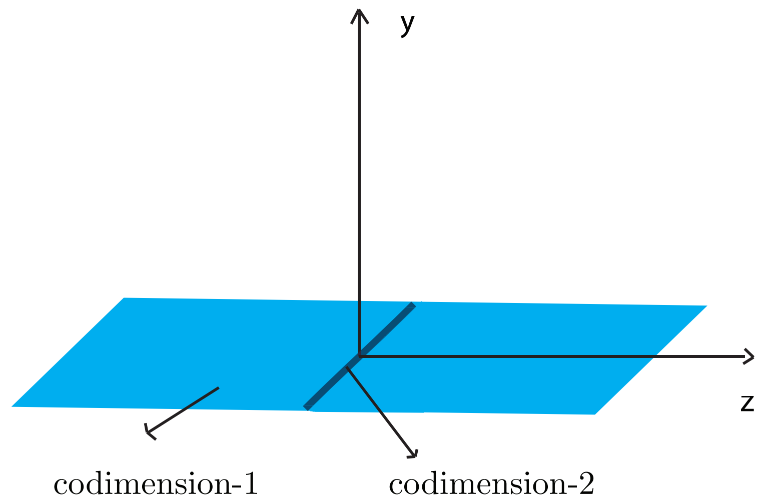

Figure 1: The configuration of the codimension-1 (plane) and codimension-2 (thick line) branes in the six-dimensional bulk is shown. The induced metric on the codimension-1 brane is g_{ab} and on the codimension-2 brane is $g_{\mu\nu}$.

where g_{ab} is the 5D induced metric on the codimension-1 brane with $a, b = z, \mu$ (here Greek letters denote the 4D space indices.). Having the bulk and brane coordinate systems, the full 6D action can be written as

$$\begin{aligned} S &= \frac{M_6^4}{2}\int d^6x\sqrt{-{}^{(6)}g}\ {}^{(6)}R + \int d^6x\sqrt{-{}^{(5)}g}\Big(\frac{M_5^3}{2}\ {}^{(5)}R + \mathcal{L}_5^{mat}\Big)\delta(y) \\ &+ \int d^6x\sqrt{-{}^{(4)}g}\Big(\frac{M_4^2}{2}\ {}^{(4)}R + \mathcal{L}_4^{mat}\Big)\delta(y)\delta(z). \end{aligned} \tag{2}$$

The codimension-1 brane is located at $y = 0$ and the codimension-2 brane is placed on the codimension-1 brane at $y = z = 0$ (see Fig. 1). We are interested in derivation of the gravitational equations on the codimension-2 brane. In the ADM formalism, the full 6D action for the cascading setup, including the appropriate Gibbons-Hawking boundary term in 6D part, can be rewritten as

$$\begin{aligned} S &= \frac{M_6^4}{2}\int d^6x\sqrt{-{}^{(6)}g}\ ({}^{(5)}R + K^2 - K_{ab}K^{ab}) + \int d^6x\sqrt{-{}^{(5)}g}\ \Big(\frac{M_5^3}{2}{}^{(5)}R + \mathcal{L}_5^{mat}\Big)\delta(y) \\ &+ \int d^6x\sqrt{-{}^{(4)}g}\ \Big(\frac{M_4^2}{2}{}^{(4)}R + \mathcal{L}_4^{mat}\Big)\delta(y)\delta(z), \end{aligned} \tag{3}$$

where the extrinsic curvature K_{ab} is given by

$$K_{ab} = \frac{1}{2N}(\partial_y g_{ab} - \nabla_a N_b - \nabla_b N_a), \tag{4}$$

and $K = K_{ab}g^{ab}$. To derive the gravitational equations we should take the variation of the action (3) with respect to the variables N, N_a and g_{ab} which vanish at infinity but are nonzero on the boundary branes. Thus the boundary terms arising in the variations give the contributions to the gravitational theory on the hypersurface of $y = 0$ (codimension-1 brane) and hypersurface of $y = z = 0$ (codimension-2 brane). In the following sections, we consider the variation of the 6D, 5D and 4D parts of the action and derive the effective gravitational theory on the codimension-1 and codimension-2 brane, separately.

2.1 Variation of the 6D part of the action

Taking the variation of the bulk terms with respect to N, N_a and g_{ab} gives the bulk Einstein equations

$$ {}^{(6)}G_{AB} = 0, \qquad A, B = y, z, \mu. \tag{5} $$

The boundary contributions in the variation of the 6D part of the action give the contributions localized to the hypersurface $y = 0$. Since there is no term coming from $\delta\partial_y N$ and $\delta\partial_y N_a$ the only terms localized to the codimension-1 brane come from the term proportional to $\delta\partial_y g_{ab}$. Since δg_{ab} does not vanish on the brane, the boundary contributions to the variations of the 6D part of the action read

$$ -\frac{M_6^4}{2}\Delta_y \int dz d^4x \sqrt{-{}^{(5)}g}\,(K g^{ab} - K^{ab})\delta g_{ab}. \tag{6} $$

Here we have used $\sqrt{-{}^{(6)}g} = N\sqrt{-{}^{(5)}g}$ and $\triangle_y$ is the discontinuity in a given quantity A over the codimension-1 brane namely, $\triangle_y A = 2A|_{y=0+}$. From now on we omit the subscript $\Big|_{y=0+}$.

2.2 Variation of the 5D part of the action

Now we take the variation of the 5D part of the action (3) with respect to g_{ab} which gives us the contribution of the 5D part on the codimension-1 brane. Since there is a codimension-2 brane embedded on the codimension-1 brane, there are also contributions localized to the hypersurface $y = z = 0$. To obtain the 5D contributions localized to the codimension-2 brane, we go to the ADM form of the 5D metric g_{ab} adapted to the codimension-2 brane where z plays the role of a time variable [2]

$$ {}^{(5)}ds^2 = g_{ab}dx^a dx^b = \mathcal{N}^2 dz^2 + g_{\mu\nu}(dx^\mu + \mathcal{N}^\mu dz)(dx^\nu + \mathcal{N}^\nu dz), \tag{7} $$

where $g_{\mu\nu}$ is the 4D induced metric on codimension-2 brane. In the ADM formalism in the 5D spacetime, the 5D part of the action (2) can be rewritten as

$$ \begin{aligned} \int d^6x \sqrt{-{}^{(5)}g}\Big(\frac{M_5^3}{2}\,{}^{(5)}R + \mathcal{L}_5^{mat}\Big)\delta(y) &= \frac{M_5^3}{2}\int d^6x \sqrt{-{}^{(5)}g}({}^{(4)}R + \mathcal{K}^2 - \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu})\delta(y) \\ &+ \int d^6x \sqrt{-{}^{(5)}g}\mathcal{L}_5^{mat}\delta(y), \end{aligned} \tag{8} $$

where

$$ \mathcal{K}_{\mu\nu} = \frac{1}{2\mathcal{N}}(\partial_z g_{\mu\nu} - \nabla_\mu \mathcal{N}_\nu - \nabla_\nu \mathcal{N}_\mu), \tag{9} $$

and $\mathcal{K} = g^{\mu\nu}\mathcal{K}_{\mu\nu}$. Varing the 5D part of the action with respect to g_{ab} leads to

$$ \frac{1}{2}\int dz d^4x \sqrt{-{}^{(5)}g}(M_5^3\,{}^{(5)}G_{ab} - S_{ab})\delta g^{ab}, \tag{10} $$

which is localized to the hypersurface $y = 0$. Here S_{ab} is the energy momentum tensor for the matter on the codimension-1 brane. The contributions localized to the codimension-2 brane come from the variation $\delta\partial_z g_{\mu\nu}$. Note that there is no term which is proportional to $\delta\partial_z\mathcal{N}_\mu$ and $\delta\partial_z\mathcal{N}$ in the variation of the 5D part of the

[2] The 5D metric is

$$ g_{ab} = \begin{pmatrix} \mathcal{N}^2 + \mathcal{N}^\mu\mathcal{N}_\mu & \mathcal{N}_\mu \\ \mathcal{N}_\mu & g_{\mu\nu} \end{pmatrix}. $$

action. Thus the only boundary term localized to the codimension-2 brane coming from the variations of the 5D part of the action is

$$-\frac{M_5^3}{2}\triangle_z \int d^4x\sqrt{-{}^{(4)}g}(\mathcal{K}g^{\mu\nu}-\mathcal{K}^{\mu\nu})\delta g_{\mu\nu}. \tag{11}$$

where we have used $\sqrt{-{}^{(5)}g}=\mathcal{N}\sqrt{-{}^{(4)}g}$. The discontinuity in a given quantity A across the hypersurface of $z=0$ is given by $\triangle_z A = 2A|_{z=0+}$. From now on we omit the subscript $|_{z=0+}$.

2.3 Variation of the 4D part of the action

Finally, varying the 4D part of the action with respect to g_{ab} leads to

$$\frac{1}{2}\int d^4x\sqrt{-{}^{(4)}g}(M_4^2\ {}^{(4)}G_{\mu\nu}-T_{\mu\nu})\delta_a^\mu\delta_b^\nu\delta g^{ab}, \tag{12}$$

where $T_{\mu\nu}$ is the energy momentum tensor for the matter on the codimension-2 brane.

2.4 Effective gravitational equations on the branes

Collecting the boundary contributions in the variations of all parts of the action (2) derived in the previous sections, one can obtain the effective gravitational equations on the codimension-1 and codimension-2 branes. Combining (6) with (10), (11) and (12), the boundary equations on the codimension-1 brane at $y=0$ read

$$\begin{aligned}\int dzd^4x\sqrt{-{}^{(5)}g}\ (M_5^{3(5)}G_{ab}-S_{ab}) &= \Delta_y\int dzd^4x\sqrt{-{}^{(5)}g}\ M_6^4(K_{ab}-Kg_{ab}) \\ &+ \int dzd^4x\sqrt{-{}^{(4)}g}\ \Big[-M_4^{2(4)}G_{\mu\nu}+T_{\mu\nu}+2M_5^3(\mathcal{K}_{\mu\nu}-\mathcal{K}g_{\mu\nu})\Big]\delta_a^\mu\delta_b^\nu\delta(z).\end{aligned} \tag{13}$$

Now we assume that the shift vector $N_a = n_a(x)s(y)\epsilon(z)$. The function $\epsilon(z)$ is a regulating function with the following properties: $\epsilon(\infty)=1$, $\epsilon(-z)=-\epsilon(z)$ and $\epsilon(z)_{,z}=2\delta_\epsilon(z)$, where $\delta_\epsilon(z)$ is a regularization of the Dirac delta function. The function $s(y)$ is the sign function and $s(y)_{,y}=2\delta(y)$. Using this ansatz one can see that the first term on the right hand side of equation (13) includes some terms localized to the codimension-2 brane. One can see this using the components of the extrinsic curvature K_{ab}

$$\begin{aligned}K_{\mu\nu} &= \frac{1}{2N}\Big[\partial_y g_{\mu\nu}-\partial_{(\mu}N_{\nu)}+2\Gamma^a_{\mu\nu}N_a\Big]=\tilde{K}_{\mu\nu}, \\ K_{zz} &= \frac{1}{2N}\Big[\partial_y g_{zz}-4n_z(x)s(y)\delta_\epsilon(z)+2\Gamma^a_{zz}N_a\Big]=\tilde{K}_{zz}-\frac{2n_z(x)s(y)}{N}\delta_\epsilon(z), \\ K_{z\mu} &= \frac{1}{2N}\Big[\partial_y g_{z\mu}-2n_\mu(x)s(y)\delta_\epsilon(z)-\partial_\mu N_z+2\Gamma^a_{\mu z}N_a\Big]=\tilde{K}_{z\mu}-\frac{n_\mu(x)s(y)}{N}\delta_\epsilon(z),\end{aligned} \tag{14}$$

where $\tilde{K}_{ab}$ represents the extrinsic curvature except for the terms proportional to $\delta_\epsilon(z)$. Thus the gravitational equations on the codimension-1 brane at $y=0$, off the codimension-2 brane at $z=0$, are given by

$${}^{(5)}G_{ab}=\frac{1}{M_5^3}S_{ab}+\frac{2M_6^4}{M_5^3}(\tilde{K}_{ab}-\tilde{K}g_{ab}). \tag{15}$$

Finally, let us derive the gravitational equations on the codimension-2 brane at $y=z=0$. Plugging the components (14) into equation (13), and taking the integral over z across the codimension-2 brane, we obtain the localized terms to the codimension-2 brane coming from the boundary contribution to the variations of the action (2). Therefore, the boundary equations on the codimension-2 brane read

$$n_\nu(x)=0, \tag{16}$$

$$
{}^{(4)}G_{\mu\nu} = \frac{1}{M_4^2}T_{\mu\nu} + 2\frac{M_5^3}{M_4^2}(\mathcal{K}_{\mu\nu} - \mathcal{K}g_{\mu\nu}) + 4\frac{M_6^4}{M_4^2}\int dz \frac{\mathcal{N}}{N} n_z(x)\Big(g^{zz}g_{\mu\nu} - g^z{}_\mu g^z{}_\nu\Big)\delta_\epsilon(z). \tag{17}
$$

Equations (13), (16) and (17) along with the bulk Einstein equations (5) are equations of motion in the cascading gravity model. It is obvious that when $M_6^4 = 0$, the DGP gravitational equations on the codimension-2 brane are recovered.

3 Cosmology

3.1 Bulk geometry

In this section, we apply the formulation developed in the previous section to cosmology. We firstly present the bulk metric where a general Friedmann-Robertson-Walker (FRW) codimension-2 brane and a codimension-1 brane are embedded. We start from the 6D Minkowski spacetime with the foliation

$$
\begin{aligned}
ds_6^2 &= (c^2-\beta^2)dy^2 + (dz+\beta dy)^2 - \Big(1 + (\frac{\beta}{c}z + cy)\big(H + \frac{\dot{H}}{H}\big)\Big)^2 dt^2 \\
&+ a(t)^2\Big(1 + (\frac{\beta}{c}z + cy)H\Big)^2 \delta_{ij}dx^i dx^j,
\end{aligned} \tag{18}
$$

where β and c are constants, $a(t)$ is an arbitrary function of t and $H(t) := \frac{\dot{a}}{a}$. "dot" denotes the derivative with respect to t. The metric (18) clearly satisfies the 6D Einstein equation (5). Note that if β and c are time dependent the metric is not a solution to equation (5). On the $y = z = 0$ hypersurface where the metric can be written as

$$
ds_4^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \tag{19}
$$

the function $a(t)$ can be interpreted as the cosmic scale factor of the FRW Universe, where H becomes the Hubble parameter. Imposing the Z_2-symmetry across the $y = 0$ and $z = 0$ hypersurfaces and choosing $c^2 = 1 + \beta^2$, the bulk metric reads

$$
\begin{aligned}
ds_6^2 &= \Big(1 + \beta^2\big(1 - \epsilon(z)^2\big)\Big)dy^2 + (dz + \beta s(y)\epsilon(z)dy)^2 \\
&- \Big(1 + \big(\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}} + \sqrt{1+\beta^2}|y|\big)\big(H + \frac{\dot{H}}{H}\big)\Big)^2 dt^2 + a^2\Big(1 + \big(\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}} + \sqrt{1+\beta^2}|y|\big)H\Big)^2\delta_{ij}dx^i dx^j.
\end{aligned} \tag{20}
$$

From the comparison with the ADM metric in equation (1), it is straightforward to read off the lapse function, shift vector and induced metric components. It is clear that the metric (20) is invariant under the parity transformation $y \to -y$ or $z \to -z$. The assumption of regularized profile of $\epsilon(z)$ is particularly important to see the contributions from the 6D bulk on the codimension-2 brane. On the other hand, it is enough to assume the codimension 1-brane as a distributional object, where $\epsilon(z)$ and $\delta_\epsilon(z)$ are treated as the usual sign function and the delta function, respectively.

3.2 General cosmological equations on the codimension-2 brane

We now derive the cosmological solutions in the 4D spacetime. Assuming metric (20), the induced metric on the $y = 0$ hypersurface is given by

$$
ds_5^2 = dz^2 - \Big(1 + \frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}\big(H + \frac{\dot{H}}{H}\big)\Big)^2 dt^2 + a^2\Big(1 + \frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}H\Big)^2\delta_{ij}dx^i dx^j. \tag{21}
$$

Calculating the extrinsic and intrinsic curvature tensors (see Appendix B for details) and using boundary equation (17), we find the modified Friedmann equations on the codimension-2 brane

$$3M_4^2 H^2 = \rho + \rho_{\rm eff}, \tag{22}$$

$$-M_4^2\big(2\dot{H} + 3H^2\big) = p + p_{\rm eff}, \tag{23}$$

where ρ and p are energy density and pressure of the matter localized to the codimension-2 brane, and $\rho_{\rm eff}$ and $p_{\rm eff}$ represent the effective energy density and pressure of the dark component

$$\begin{aligned} \rho_{\rm eff} &:= \frac{6M_5^3\beta}{\sqrt{1+\beta^2}}H - 4M_6^4\int dz \frac{\beta\delta_\epsilon(z)}{\sqrt{1+\beta^2(1-\epsilon(z)^2)}}\Big(1+\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}\big(\frac{\dot{H}}{H}+H\big)\Big)^2, \\ p_{\rm eff} &:= -\frac{6M_5^3\beta}{\sqrt{1+\beta^2}}\Big(H+\frac{\dot{H}}{3H}\Big) + 4M_6^4\int dz \frac{\beta\delta_\epsilon(z)}{\sqrt{1+\beta^2(1-\epsilon(z)^2)}}\Big(1+\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}H\Big)^2. \end{aligned} \tag{24}$$

The effective equation of state is given by

$$\begin{aligned} w_{\rm eff} &:= \frac{p_{\rm eff}}{\rho_{\rm eff}} \\ &= -1 - \frac{\dot{H}}{H}\frac{1}{\rho_{\rm eff}}\Big[\frac{2M_5^3\beta}{\sqrt{1+\beta^2}} + 4M_6^4\int dz \frac{\beta\delta_\epsilon(z)}{\sqrt{1+\beta^2(1-\epsilon(z)^2)}}\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}\Big(2+\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}\Big(2H+\frac{\dot{H}}{H}\Big)\Big)\Big]. \end{aligned} \tag{25}$$

Finally, the Bianchi identity on the codimension-2 brane gives the nonconservation law

$$\begin{aligned} \dot{\rho} + 3H\big(\rho+p\big) &= 4M_6^4\int dz \frac{\beta\delta_\epsilon(z)}{\sqrt{1+\beta^2(1-\epsilon(z)^2)}}\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}\Big\{2\frac{d}{dt}\Big(H+\frac{\dot{H}}{H}\Big)\Big(1+\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}\Big(H+\frac{\dot{H}}{H}\Big)\Big) \\ &+ 3\dot{H}\Big(2+\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}\big(2H+\frac{\dot{H}}{H}\big)\Big)\Big\}. \end{aligned} \tag{26}$$

This may not be a surprising fact since the inclusion of a codimension-2 brane thickness is already somewhat beyond our theory (2) where the codimension-2 brane is assumed to be an infinitesimally thin object. In other words, to see the nontrivial bulk effect on the brane dynamics, it is necessary to include a finite brane thickness. For a de Sitter or Minkowski codimension-2 brane where H is constant the energy conservation law on the codimension-2 brane is satisfied even including a finite thickness.

From now on, in the main text of this paper we will focus on the cosmological dynamics on the codimension-2 brane. Concerning the dynamics on the codimension-1 brane, we ask readers to see the Appendix A.

3.3 Small brane thickness approximation

In general it is impossible to perform the integrals in equation (24) analytically. We here use an approximation in which the codimension-2 brane thickness, σ, is much smaller than the size of the cosmic horizon H^{-1}. The details of the approximation are shown in Appendix C. We then find

$$\begin{aligned} \rho_{\rm eff} &= \frac{6M_5^3\beta}{\sqrt{1+\beta^2}}H - 4M_6^4\Big[\arctan(\beta) + \sigma C(\beta)\Big(H+\frac{\dot{H}}{H}\Big)\Big], \\ p_{\rm eff} &= -\frac{6M_5^3\beta}{\sqrt{1+\beta^2}}\Big(H+\frac{\dot{H}}{3H}\Big) + 4M_6^4\Big[\arctan(\beta) + \sigma C(\beta)H\Big], \end{aligned} \tag{27}$$

where we have defined

$$C(\beta) := -\frac{1}{\sqrt{1+\beta^2}} + \frac{\sqrt{1+\beta^2}}{\beta}\arctan(\beta), \tag{28}$$

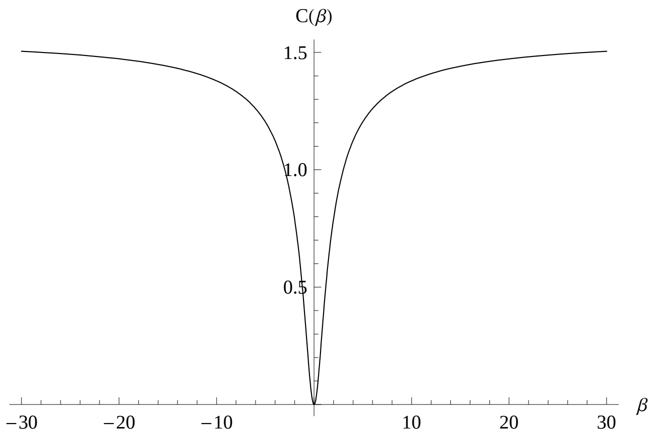

Figure 2: $C(\beta)$ defined in equation (28)

which is a non-negative even function of β, vanishes at $\beta = 0$ and monotonically increases toward $\frac{\pi}{2}$ as $|\beta| \to \infty$. The conservation law on the codimension-2 brane equation (26) reduces to

$$\dot{\rho} + 3H(\rho + p) = 4M_6^4 \sigma C(\beta) \Big[\frac{d}{dt} \Big(\frac{\dot{H}}{H} \Big) + 4\dot{H} \Big]. \tag{29}$$

Thus, due to a finite thickness of the codimension-2 brane there is always an energy exchange between the bulk and brane. The effective equation of state (26) becomes

$$w_{\text{eff}} = -1 - \frac{\dot{H}}{H^2} \frac{\frac{2M_5^3 \beta}{\sqrt{1+\beta^2}} + 4M_6^4 \sigma C(\beta)}{\frac{6M_5^3 \beta}{\sqrt{1+\beta^2}} - \frac{4M_6^4}{H} \big(\arctan(\beta) + \sigma C(\beta) \big(H + \frac{\dot{H}}{H} \big) \big)}. \tag{30}$$

For $\big|\dot{H}\big| \ll H^2$ and since $\sigma H \ll 1$,

$$w_{\text{eff}} \simeq -1 - \frac{\dot{H}}{H^2} \frac{\frac{M_5^3 \beta}{\sqrt{1+\beta^2}} + 2M_6^4 \sigma C(\beta)}{\frac{3M_5^3 \beta}{\sqrt{1+\beta^2}} - \frac{2M_6^4}{H} \arctan(\beta)}. \tag{31}$$

4 De Sitter solutions and the features of degravitation

In this section, we give the solutions with Minkowski and de Sitter codimension-2 branes in the cascading model. In particular, we discuss the possible features of degravitation.

4.1 Minkowski codimension-2 brane

The simplest solution is a Minkowski codimension-2 brane solution which is realized by setting $H = 0$ and $\dot{H} = 0$ in (20). In this case, the 6D bulk metric is given by

$$ds_6^2 = \Big(1 + \beta^2 \big(1 - \epsilon(z)^2\big)\Big) dy^2 + (dz + \beta s(y) \epsilon(z) dy)^2 - dt^2 + \delta_{ij} dx^i dx^j. \tag{32}$$

The codimension-2 brane is supported by the tension $\rho = -p = \lambda$. Modified Friedmann equations (22) reduce to the single equation which relates the brane tension to the bulk geometry

$$\lambda = 4M_6^4 \arctan(\beta), \tag{33}$$

where the left-hand side shows the deficit angle in the bulk. For a Minkowski codimension-2 brane, $\tilde{K}_{ab} = 0$ and the codimension-1 brane geometry is also the 5D Minkowski spacetime. Since $|\arctan(\beta)| < 1$, the brane tension can be at most of order M_6^4. In Refs. [4, 6], it has been shown that the cascading gravity is ghost free if the codimension-2 brane tension satisfies the following bound

$$\lambda > \frac{2r_3}{3r_4} M_6^4. \tag{34}$$

This condition is satisfied as long as two cross-over scales satisfy $r_3 < r_4$. Perturbations about the Minkowski codimension-2 brane solution (32) and stability have been explicitly analyzed in the recent work Ref. [6].

4.2 De Sitter codimension-2 brane and degravitation

Now we consider the de Sitter codimension-2 brane solution where $a(t) = a_0 e^{H_0 t}$. In particular, we focus on the possible features of degravitation. In this case the 6D bulk metric (20) reduces to

$$\begin{aligned} ds_6^2 &= \Big(1 + \beta^2 \big(1 - \epsilon(z)^2\big)\Big) dy^2 + (dz + \beta s(y)\epsilon(z) dy)^2 - \Big(1 + \big(\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}} + \sqrt{1+\beta^2}|y|\big) H_0\Big)^2 dt^2 \\ &+ a_0^2 e^{2H_0 t} \Big(1 + \big(\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}} + \sqrt{1+\beta^2}|y|\big) H_0\Big)^2 \delta_{ij} dx^i dx^j. \end{aligned} \tag{35}$$

Here the codimension-2 brane can be supported only by the tension with $\rho = -p = \lambda$. The effective cosmological equations (22) reduce to a single integral equation

$$3M_4^2 H_0^2 = \lambda + \rho_{\rm eff}, \tag{36}$$

where

$$\rho_{\rm eff} = -p_{\rm eff} = \frac{6M_5^3\beta}{\sqrt{1+\beta^2}} H_0 - 4M_6^4 \int dz \frac{\beta\delta_\epsilon(z)}{\sqrt{1+\beta^2\big(1-\epsilon(z)^2\big)}} \Big(1 + \frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}} H_0\Big)^2. \tag{37}$$

In the following, we are going to obtain explicit de Sitter 3-brane solutions. The solution to equation (36) gives the expansion rate of the de Sitter codimension-2 brane. The solution of $H_0 > 0$ (< 0) represents an expanding (contracting) de Sitter Universe in terms of the flat slicing. In the case of $\lambda = 0$, we discuss some self-accelerating solutions.

4.2.1 Recovering DGP solutions

In the absence of the bulk gravity where $M_6 \to 0$, the modified cosmological equation on codimension-2 brane reads

$$3M_4^2 H_0^2 = \lambda + \frac{6M_5^3\beta}{\sqrt{1+\beta^2}} H_0, \tag{38}$$

which corresponds to cosmological equation in DGP model with a codimension-2 brane tension. The solution is then given by

$$H_0^{(\pm)} = \frac{\frac{3M_5^3\beta}{\sqrt{1+\beta^2}} \pm \sqrt{\big(\frac{3M_5^3\beta}{\sqrt{1+\beta^2}}\big)^2 + 3M_4^2\lambda}}{3M_4^2}, \tag{39}$$

where $H_0^{(+)} > H_0^{(-)}$. If $\lambda > 0$, irrespective of sign of β the solution of $H_0^{(+)}$ represents the expanding de Sitter Universe and that of $H_0^{(-)}$ represents the contracting de Sitter Universe. If $-\frac{3M_5^6\beta^2}{(1+\beta^2)M_4^2} < \lambda < 0$, for $\beta > 0$ both

λ	β	$H_0^{(+)}$	$H_0^{(-)}$
$\lambda > 0$	$\beta > 0$	Expanding	Contracting
	$\beta < 0$	Expanding	Contracting
$\lambda = 0$	$\beta > 0$	Expanding	Minkowski
	$\beta < 0$	Minkowski	Contracting
$-\frac{3M_5^6\beta^2}{M_4^2(1+\beta^2)} < \lambda < 0$	$\beta > 0$	Expanding	Expanding
	$\beta < 0$	Contracting	Contracting

Table 1: In this Table the classification of solutions is shown. The terms "expanding" and "contracting" denote the expanding and contracting de Sitter Universe, respectively. The term "Minkowski" denotes the Minkowski codimension-2 brane. Note that the same terms are used in the subsequent tables.

the solutions of $H_0^{(\pm)}$ represent the expanding de Sitter Universe while for $\beta < 0$ both the solutions of $H_0^{(\pm)}$ represent the contracting de Sitter Universe. If $\lambda = 0$, for $\beta > 0$ the solution of $H_0^{(+)} = \frac{2M_5^3\beta}{\sqrt{1+\beta^2}M_4^2}$ gives the self-accelerating solution of the DGP model, while for $\beta < 0$ the solution of $H_0^{(+)} = 0$ gives the normal branch Minkowski brane solution of DGP. But the self-accelerating solution in the DGP model is known to be unstable. The classification of solutions is shown in Table 1.

4.2.2 Degravitation features in the limit of zero brane thickness

In this section we study the effective cosmological equations (36) in the limit of zero thickness of the codimension-2 brane. In this case equation (36) becomes

$$3M_4^2H_0^2 = \tilde{\lambda} + \frac{6M_5^3\beta}{\sqrt{1+\beta^2}}H_0, \quad \tilde{\lambda} := \lambda - 4M_6^4\arctan(\beta). \tag{40}$$

The solution is simply given by equation (39) classified in Table 1, with replacement of λ with $\tilde{\lambda}$. In the case of $\lambda = 0$, for $\beta > 0$ both the solutions of $H_0^{(\pm)}$ give rise to a self-accelerating Universe for $M_5^6 > \frac{4M_4^2M_6^4(1+\beta^2)\arctan\beta}{3\beta^2}$. For $\beta < 0$ only the solution of $H_0^{(+)}$ provides the self-accelerating Universe, while $H_0^{(-)}$ leads to a contracting Universe. However, these self-accelerating solutions might be unstable against perturbations due to the possible existence of the ghost mode, because they do not satisfy the condition (34).

Now we argue the possible connections of our solutions with the degravitation expected in the cascading model. We focus on the case of $\tilde{\lambda} < 0$ and $\beta > 0$ where both solutions represent the expanding de Sitter Universe. In the limiting case of $3M_5^6\beta^2 \gg -(1+\beta^2)M_4^2\tilde{\lambda}$, we have

$$\frac{H_0^{(-)}}{H_0^{(+)}} \simeq -\frac{(1+\beta^2)}{12\beta^2}\frac{r_3}{r_4}\frac{\tilde{\lambda}}{M_6^4} \ll 1, \tag{41}$$

for $\frac{\tilde{\lambda}}{M_6^4} = O(1)$ and as long as $r_3 \ll r_4$. The physical meaning of equation (41) is clearly understood as follows: The $H_0^{(-)}$ solution with the tension $\frac{2}{3}\frac{M_4^2M_6^8}{M_5^6} < \lambda < 4M_6^4\arctan\beta$, which is expected to be ghost free, gives a much smaller expansion rate than one in the self-accelerating branch in the 5D DGP model with an expansion rate of order $H_0^{(+)}$. In addition, rewriting the $H_0^{(-)}$ solution in the same limit

$$H_0^{(-)2} \simeq \frac{1}{3M_4^2}\left(\frac{r_3}{r_4}\right)|\tilde{\lambda}| \ll \frac{1}{3M_4^2}|\tilde{\lambda}|, \tag{42}$$

shows that the effect of the effective vacuum energy $\tilde{\lambda}$ into the Hubble expansion rate is suppressed by the ratio $\frac{r_3}{r_4}$ in comparison with the ordinary 4D cosmology. This implies a deep connection of

our results with the degravitation idea. This result shows that cascading model can provide a mechanism that could support a small expansion rate in the presence of a large cosmological constant i.e., tension. Moreovere, although the suppression in the 6D model is not enough to explain the fine-tuning problem for obtaining a tiny expansion rate which is suggested from observations, the cascading gravity model may be extendable to higher dimensional cases, which may lead to more cross-over scales $r_i := \frac{M_{i+1}^{i-1}}{M_{i+2}^{i}}$ $(i = 5, 6, 7, \cdots)$ with $r_i \ll r_{i+1}$. Then we could expect that the fine-tuning problem is alleviated more than in the 6D model.

4.2.3 Degravitation features with a small brane thickness

In general, it is impossible to solve the integral equation (36) with (37) analytically when a finite thickness of the codimension-2 brane is taken into consideration. Here, we use the small thickness approximations given in the previous section and Appendix C. Using (27), the equation (36) reduces to the quadratic equation for H_0 which leads to the solutions

$$\begin{aligned} H_0 = H_0^{(\pm)} \quad &:= \quad \frac{1}{3\sqrt{1+\beta^2}M_4^2}\Big[3M_5^3\beta - 2\sqrt{1+\beta^2}\sigma C(\beta)M_6^4 \\ &\pm \quad \sqrt{\left(3M_5^3\beta - 2\sqrt{1+\beta^2}\sigma C(\beta)M_6^4\right)^2 + 3M_4^2(1+\beta^2)\tilde{\lambda}}\Big]. \end{aligned} \tag{43}$$

There is no physical solution for

$$\tilde{\lambda} < \tilde{\lambda}_* := -\frac{(3M_5^3\beta - 2\sqrt{1+\beta^2}\sigma C(\beta)M_6^4)^2}{3(1+\beta^2)M_4^2}. \tag{44}$$

The solutions are classified in Table 3. If $\tilde{\lambda} > 0$ the solution of $H_0^{(+)}$ always represents the expanding de Sitter Universe. If $\tilde{\lambda}_* < \tilde{\lambda} < 0$ and $\beta > \frac{2\sqrt{1+\beta^2}\sigma C(\beta)M_6^4}{3M_5^3}$ both the solutions of $H_0^{(\pm)}$ represent the expanding de Sitter Universe. In the absence of the brane tension, $\lambda = 0$, we can obtain self-accelerating solutions. If $\beta > 0$, for $3M_5^3\beta > 2\sqrt{1+\beta^2}\sigma C(\beta)M_6^4$, both the solutions of $H_0^{(\pm)}$ give the self-accelerating universes, while if $\beta < 0$ only the solution of $H_0^{(+)}$ gives the self-accelerating universe. However, these self-accelerating solutions might be unstable against perturbations since they do not satisfy the condition (34).

Finally, we argue the connections of the $H_0^{(-)}$ solution with the idea of degravitation. For $\beta > 0$ and $3M_5^3\beta > 2\sqrt{1+\beta^2}\sigma C(\beta)M_6^4$, in the limit of $\tilde{\lambda} \gg \tilde{\lambda}_*$, we obtain

$$\frac{H_0^{(-)}}{H_0^{(+)}} \simeq -\frac{1}{\left(1 - \frac{2\sqrt{1+\beta^2}\sigma C(\beta)}{3\beta r_4}\right)^2}\frac{1+\beta^2}{12\beta^2}\left(\frac{r_3}{r_4}\right)\frac{\tilde{\lambda}}{M_6^4} \ll 1, \tag{45}$$

and

$$H_0^{(-)2} \simeq \frac{1}{3M_4^2}\left(\frac{r_3}{r_4}\right)\frac{1}{\left(1 - \frac{2\sqrt{1+\beta^2}\sigma C(\beta)}{3\beta r_4}\right)^2}|\tilde{\lambda}|, \tag{46}$$

for $\frac{\tilde{\lambda}}{M_6^4} = O(1)$ and $\frac{r_3}{r_4} \ll 1$. This equation differs from equation (42) by the factor $\left(1 - \frac{2\sqrt{1+\beta^2}\sigma C(\beta)}{3\beta r_4}\right)^{-2}$which depends on the thickness of the brane. Since $r_4 \gg \sigma$, where r_4 could be macroscopic and σ is microscopic, the effects of the brane thickness are small. Therefore, one could still conclude that the Hubble expansion rate is much smaller than the one in ordinary 4D cosmology, namely $H_0^{(-)2} \ll \frac{1}{3M_4^2}|\tilde{\lambda}|$. Thus, irrespective of the inclusion of a small thickness, the cascading model exhibits features of degravitation.

λ	β	$H_0^{(+)}$	$H_0^{(-)}$
$\tilde{\lambda} > 0$	$\beta > \frac{2\sqrt{1+\beta^2}\sigma C M_6^4}{3M_5^3}$	Expanding	Contracting
	$\beta < \frac{2\sqrt{1+\beta^2}\sigma C M_6^4}{3M_5^3}$	Expanding	Contracting
$\tilde{\lambda} = 0$	$\beta > \frac{2\sqrt{1+\beta^2}\sigma C M_6^4}{3M_5^3}$	Expanding	Minkowski
	$\beta < \frac{2\sqrt{1+\beta^2}\sigma C M_6^4}{3M_5^3}$	Minkowski	Contracting
$\tilde{\lambda}_* < \tilde{\lambda} < 0$	$\beta > \frac{2\sqrt{1+\beta^2}\sigma C M_6^4}{3M_5^3}$	Expanding	Expanding
	$\beta < \frac{2\sqrt{1+\beta^2}\sigma C M_6^4}{3M_5^3}$	Contracting	Contracting

Table 2: Classification of solutions with a codimension-2 brane thickness.

5 Conclusion

We have presented a formulation of the nonlinear effective gravitational theory in the 6D cascading gravity model. This model is a higher-dimensional extension of the 5D DGP braneworld model. In the simplest 6D model we are living on the codimension-2 brane that is located on a codimension-1 brane embedded into a 6D bulk. The particularly interesting expectations are that this model may exhibit a degravitation where the gravitational force falls off sufficiently fast, and is also free from a ghost instability if the tension of the codimension-2 brane satisfies a bound. An important aim for presenting our formulation is to see whether in reality the idea of degravitation could work for the cascading model, through an explicit investigation of cosmological solutions. The gravitational equations on the codimension-2 brane are composed of the contributions of the matter on the codimension-2 brane, the induced gravity on the codimension-1 brane and the gravity in the 6D bulk. The bulk contribution is given by integrating over the sixth direction across the codimension-2 brane along the codimension-1 brane. After the derivation of the general equations of motion in the cascading model, we have applied them to cosmology where the codimension-2 brane geometry is described by a flat Friedmann-Robertson-Walker Universe, and obtained the modified Friedmann equations. In the zero thickness limit of the codimension-2 brane, the bulk contribution becomes an effective cosmological constant whose sign depends on the sign of β. A finite thickness, however, leads to an energy exchange between the bulk and codimension-2 brane, except for the cases where the codimension-2 brane geometry is exactly Minkowski or de Sitter. On the other hand, there is no energy exchange from or into the codimension-1 brane in any case (see Appendix A).

Finally, we have discussed the Minkowski and de Sitter codimension-2 brane solutions. The Minkowski codimension-2 brane solution is realized if both the bulk and codimension-1 brane are empty and the codimension-2 brane tension takes a particular value determined by the bulk geometry. In the de Sitter brane solutions, the bulk gravity effect gives rise to a new branch of the solution which can give an expanding de Sitter codimension-2 brane solution expected to be stable and has a much smaller expansion rate than in the original DGP model, which leads to alleviation of the fine-tuning problem. We have also shown some de Sitter solutions with the small expansion rate which could have deep connections with the idea of degravitation. This result shows that the cascading model provides a dynamical mechanism to resolve the cosmological constant problem.

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A Dynamics on the codimension-1 brane

Here we shall briefly mention the dynamics on the codimension-1 brane. The nonvanishing components of the extrinsic curvature tensors on the codimension-1 brane are given in the Appendix B. The components of the energy-momentum tensor on the codimension-1 brane S_{ab} are determined by the gravitational equations (13). From equation (13) and the fact of $\tilde{K}_{z\mu} = 0$ along the codimension-1 brane ($\mu = t, i$), we get $S_{z\mu}$=0. Thus, it is straightforward to check that the components of the extrinsic curvature tensor satisfy the condition $\nabla_b(\tilde{K}^b_a - \delta^b_a\tilde{K}) = 0$. Hence, the conservation law equation $\nabla_b S^b{}_a = 0$ is also satisfied. Similarly in the case of a codimension-2 brane without a thickness, since all the components of the 6D Einstein tensor vanish everywhere in the bulk, there is no energy exchange between the codimension-2 brane and the bulk. But an inclusion of a finite thickness leads to an energy exchange between them if the codimension-2 brane geometry is not Minkowski or de Sitter symmetry.

For a de Sitter codimension-2 brane, using equation (13) together with (56) and (57), the energy-momentum tensor of the matter localized to the codimension-1 brane is given by

$$\begin{aligned} S^\mu{}_\nu &= \frac{-3H_0^2 M_5^3 + 6M_6^4 H_0\big(\sqrt{1+\beta^2} + \beta H_0 z\big)}{\big(\sqrt{1+\beta^2} + \beta H_0 z\big)^2}\delta^\mu{}_\nu, \\ S^z{}_z &= \frac{-6H_0^2 M_5^3 + 8M_6^4 H_0\big(\sqrt{1+\beta^2} + \beta H_0 z\big)}{\big(\sqrt{1+\beta^2} + \beta H_0 z\big)^2}. \end{aligned} \tag{47}$$

Thus the codimension-1 brane can be supported by the matter which has anisotropic pressure $S^z{}_z \neq \frac{1}{4}S^\mu{}_\mu$. Note that they are regular in the codimension-2 brane limit of $z \to 0$.

B Components of tensors

B.1 For a general cosmological brane

On the codimension-1 brane, using metric (20), we obtain the components of the extrinsic curvature $\tilde{K}_{ab}$ except for the contribution of the codimension-2 brane are given by

$$\begin{aligned} \tilde{K}^t{}_t &= \frac{1}{N}\frac{s(y)\big(H + \frac{\dot{H}}{H}\big)}{\sqrt{1+\beta^2} + \big(\beta\epsilon(z)z + (1+\beta^2)|y|\big)\big(H + \frac{\dot{H}}{H}\big)} \\ \tilde{K}^i{}_j &= \frac{\delta^i{}_j}{N}\frac{s(y)H}{\sqrt{1+\beta^2} + \big(\beta\epsilon(z)z + (1+\beta^2)|y|\big)H}, \\ \tilde{K}_{zz} &= \tilde{K}_{zt} = \tilde{K}_{zi} = 0. \end{aligned} \tag{48}$$

In the limit of $y \to 0+$, we obtain the nonvanishing components of the extrinsic curvature tensor except for the contribution of the codimension-2 brane

$$\tilde{K}^t{}_t \to \frac{H + \frac{\dot{H}}{H}}{\sqrt{1+\beta^2} + \beta z\big(H + \frac{\dot{H}}{H}\big)}, \quad \tilde{K}^i{}_j \to \frac{H}{\sqrt{1+\beta^2} + \beta z H}\delta^i{}_j, \tag{49}$$

and hence the nonvanishing components of the combination appearing in the junction condition

$$
\begin{aligned}
\tilde{K}^t{}_t - \tilde{K} &\to -\frac{3H}{\sqrt{1+\beta^2}+\beta z H}, \quad \tilde{K}^i{}_j - \delta^i{}_j \tilde{K} \to -\Big(\frac{H+\frac{\dot{H}}{H}}{\sqrt{1+\beta^2}+\beta z\big(H+\frac{\dot{H}}{H}\big)} + \frac{2H}{\sqrt{1+\beta^2}+\beta z H}\Big)\delta^i{}_j, \\
\tilde{K}^z{}_z - \tilde{K} &\to -\Big(\frac{H+\frac{\dot{H}}{H}}{\sqrt{1+\beta^2}+\beta z\big(H+\frac{\dot{H}}{H}\big)} + \frac{3H}{\sqrt{1+\beta^2}+\beta z H}\Big).
\end{aligned} \tag{50}
$$

Given the 5D metric (21), the nonvanishing components of the extrinsic curvature tensor on the codimension-1 brane ($y=0$) are given by

$$
\begin{aligned}
\mathcal{K}^t{}_t &= \Big(1+\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}\big(H+\frac{\dot{H}}{H}\big)\Big)\frac{\beta}{\sqrt{1+\beta^2}}\big(H+\frac{\dot{H}}{H}\big)\epsilon(z), \\
\mathcal{K}^i{}_j &= \delta^i{}_j\Big(1+\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}H\Big)\frac{\beta}{\sqrt{1+\beta^2}}H\epsilon(z), \quad \mathcal{K}_{ti}=0.
\end{aligned} \tag{51}
$$

In the codimension-2 brane limit of $z \to 0+$

$$
\mathcal{K}^t{}_t \to \frac{\beta}{\sqrt{1+\beta^2}}\big(H+\frac{\dot{H}}{H}\big), \quad \mathcal{K}^i{}_j \to \delta^i{}_j \frac{\beta}{\sqrt{1+\beta^2}}H. \tag{52}
$$

The components of the five-dimensional Einstein tensor on the codimension-1 brane are given by

$$
\begin{aligned}
{}^{(5)}G^z{}_z &= \frac{3(A^2-1)H}{\big(1+HAz\big)^2(H+(H^2+\dot{H})Az)}\Big(2H(H^2+\dot{H})Az+2H^2+\dot{H}\Big), \\
{}^{(5)}G^t{}_t &= \frac{3(A^2-1)H^2}{\big(1+HAz\big)^2}, \\
{}^{(5)}G^i{}_j &= \frac{H(A^2-1)}{\big(1+HAz\big)^2(H+(H^2+\dot{H})Az)}\Big(3H(H^2+\dot{H})Az+2\dot{H}+3H^2\Big)\delta^i{}_j,
\end{aligned} \tag{53}
$$

where we have defined $A := \frac{\beta}{\sqrt{1+\beta^2}}$.

On the codimension-2 brane, the nonvanishing components of Einstein tensor are given by

$$
{}^{(4)}G^t{}_t = -3H^2, \quad {}^{(4)}G^i{}_j = -\delta^i{}_j\big(3H^2+2\dot{H}\big). \tag{54}
$$

The homogeneity and isotropy of the geometry on the codimension-2 brane restricts the form of the energy-momentum tensor to have only the diagonal components

$$
T^t{}_t = -\rho, \quad T^i{}_j = p\delta^i{}_j. \tag{55}
$$

B.2 For a de Sitter codimension-2 brane

For the de Sitter codimension-2 brane the equations (50) reduce to

$$
\tilde{K}^\mu{}_\nu - \delta^\mu{}_\nu\tilde{K} = -\frac{3H_0}{\sqrt{1+\beta^2}+\beta z H_0}\delta^\mu{}_\nu, \quad \tilde{K}^z{}_z - \tilde{K} = -\frac{4H_0}{\sqrt{1+\beta^2}+\beta z H_0}, \tag{56}
$$

where H_0 is given by the solution of equation (40). On the other hand, from equation (53)

$$
{}^{(5)}G^\mu{}_\nu = -\frac{3H_0^2}{\big(\sqrt{1+\beta^2}+\beta H_0 z\big)^2}\delta^\mu{}_\nu, \quad {}^{(5)}G^z{}_z = -\frac{6H_0^2}{\big(\sqrt{1+\beta^2}+\beta H_0 z\big)^2}. \tag{57}
$$

C Small thickness approximations

Here we explain the small thickness approximation which has been used in the text. We use the representation $\epsilon(z) = \tanh\left(\frac{z}{\sigma}\right)$, which leads to $\delta_\epsilon(z) = \frac{1}{2}\epsilon'(z) = \frac{1}{2\sigma\cosh^2\left(\frac{z}{\sigma}\right)}$. In the limits of $\sigma \to 0$, they approach the usual sign $s(z)$ and delta functions $\delta(z)$, respectively. Thus σ denotes the thickness of the codimension-2 brane. Here we assume $\sigma H \ll 1$, namely the brane thickness is much smaller than the size of the cosmological horizon, which is reasonable. Keeping the leading order corrections due to a finite thickness, for example the integral term in the effective dark energy density equation (24) becomes

$$\int_{-\infty}^{\infty} dz \frac{\beta\delta_\epsilon(z)}{\sqrt{1+\beta^2(1-\epsilon(z)^2)}}\left[1+\frac{\beta\epsilon(z)z}{\sqrt{1+\beta^2}}\left(H+\frac{\dot{H}}{H}\right)\right]^2 = \frac{1}{2}\int_{-1}^{1}\frac{\beta d\epsilon}{\sqrt{1+\beta^2(1-\epsilon^2)}}\left[1+\frac{\beta\epsilon z(\epsilon)}{\sqrt{1+\beta^2}}\left(H+\frac{\dot{H}}{H}\right)\right]^2$$
$$\simeq \frac{1}{2}\int_{-1}^{1}\frac{\beta d\epsilon}{\sqrt{1+\beta^2(1-\epsilon^2)}}\left[1+\frac{2\beta\sigma\epsilon^2}{\sqrt{1+\beta^2}}\left(H+\frac{\dot{H}}{H}\right)\right] \simeq \arctan(\beta) + \left(H+\frac{\dot{H}}{H}\right)\sigma C(\beta), \tag{58}$$

where $C(\beta)$ is defined in equation (28) and shown in Fig. 1. The same procedure is also applied for the integral term in the effective pressure equation (24).

References

[1] G. Dvali, S. Hofmann and J. Khoury, Phys. Rev. D **76** 084006 (2007).

[2] C. de Rham, S. Hofmann, J. Khoury and A. J. Tolley, JCAP 0802 (2008)011.

[3] C. de Rham, Can. J. Phys.**87** 201(2009)

[4] C. de Rham, G. Dvali, S. Hofmann, J. Khoury, O. Pujolas, M. Redi and A. J. Tolley, Phys. Rev. Lett. **100**, 251603 (2008) [arXiv:0711.2072 [hep-th]].

[5] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485 (2000) 208.

[6] C. de Rham, J. Khoury and A. J. Tolley, Phys. Rev. D **81**, 124027 (2010) [arXiv:1002.1075 [hep-th]].

[7] N. Agarwal, R. Bean, J. Khoury and M. Trodden, Phys. Rev. D **81**, 084020 (2010) [arXiv:0912.3798 [hep-th]].

[8] N. Chow and J. Khoury, Phys. Rev. D **80**, 024037 (2009) [arXiv:0905.1325 [hep-th]].

[9] N. Agarwal, R. Bean, J. Khoury and M. Trodden, arXiv:1102.5091 [hep-th].

[10] J. Khoury and M. Wyman, Phys. Rev. D **80**, 064023 (2009) [arXiv:0903.1292 [astro-ph.CO]].

[11] M. Wyman and J. Khoury, Phys. Rev. D **82**, 044032 (2010) [arXiv:1004.2046 [astro-ph.CO]].